# Exercise 1

Use residues to establish the following integration formula:

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \frac{2\pi}{3}.$$

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### Solution

Because the integral goes from 0 to $2\pi$ and the integrand is in terms of $\sin\theta$, we can make the substitution, $z = e^{i\theta}$. Euler's formula states that $e^{i\theta} = \cos\theta + i\sin\theta$, so we can write $\sin\theta$ and $d\theta$ in terms of $z$ and $dz$, respectively.

$$\sin\theta = \frac{z - z^{-1}}{2i} \quad \text{and} \quad d\theta = \frac{dz}{iz}.$$

The integral becomes

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} &= \int_C \frac{1}{5 + 4\left(\frac{z - z^{-1}}{2i}\right)} \frac{dz}{iz} \\
&= \int_C \frac{dz}{5iz + 2z\left(z - z^{-1}\right)} \\
&= \int_C \frac{dz}{2z^2 + 5iz - 2} \\
&= \int_C \frac{dz}{(z + 2i)(2z + i)} \\
&= \int_C \frac{dz}{2(z + 2i)\left(z + \frac{i}{2}\right)} \\
&= \int_C f(z)\, dz,
\end{aligned}$$

where the contour $C$ is the positively oriented unit circle centered at the origin.

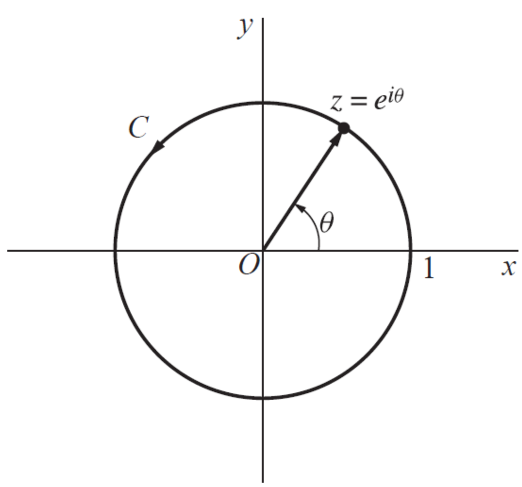


Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

According to Cauchy's residue theorem, this contour integral is $2\pi i$ times the sum of the residues of $f(z)$ at the singular points inside the contour.

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

$f(z)$ has two singular points, $z = -2i$ and $z = -\frac{i}{2}$. Since $z = -2i$ lies outside the unit circle, it makes no contribution to the integral. However, $z = -\frac{i}{2}$ does lie inside the circle, so we have to evaluate the residue of $f(z)$ at this point. Because $z = -\frac{i}{2}$ is a simple pole, the residue can be written as

$$\operatorname*{Res}_{z=-\frac{i}{2}} f(z) = \phi\left(-\frac{i}{2}\right),$$

where $\phi(z)$ is determined from $f(z)$.

$$f(z) = \frac{\phi(z)}{z + \frac{i}{2}} \quad \to \quad \phi(z) = \frac{1}{2(z+2i)}$$

So

$$\operatorname*{Res}_{z=-\frac{i}{2}} f(z) = \frac{1}{2(-\frac{i}{2} + 2i)} = \frac{1}{3i}.$$

This means that

$$\int_C f(z)\,dz = 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}.$$

And therefore,

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \frac{2\pi}{3}.$$